Discussion of the Two-Fermion Sector in a Unified Nonlinear Spinor Field Model with Indefinite Metric I

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In unified field models all observable (elementary and non-elementary) particles are assumed to be bound states of elementary unobservable fermion fields. Such models are formulated by self-regularizing higher order nonlinear spinor field equations with indefinite metric. The latter needs a careful investigation of the corresponding state space, in particular with respect to bound states. Based on preceding papers the general analysis of the state space is further developed in the framework of a relativistic energy representation in Part I. In Part II this formalism is applied to bound states of the two-fermion sector for a simple model. By direct calculation it turns out that for very heavy masses of the constituent fields bound states with positive norm and small masses are possible, *i.e.*, that the two-fermion sector allows a meaningful physical interpretation.

Introduction

Unified field models are defined by quantum field theories in which all observable (elementary and non-elementary) particles are assumed to be bound states of elementary fermion fields. Accordingly, unified field models must be formulated by dynamical laws for selfcoupled fermion fields. If the dynamical law for such selfcoupled fermion fields is given by a first order nonlinear spinor field equation with local interactions, the corresponding quantum field theory is non-renormalizable. A way out of this difficulty is possible if selfregularizing spinor field equations are used. The selfregularization property is provided by higher order field equations. This property was discovered by Bopp [1] in constructing a selfregularizing classical electrodynamics. Wildermuth [2] first discussed higher order linear spinor field equations. A short survey of the further development is given in [3]. Unified field models of matter which are based on higher order nonlinear spinor field equations and which are intended to give an explanation for the existence of the hierarchy of elementary particles and forces have recently been proposed by Grosser and Lauxmann [4] for fractional charges and by the author [5] for integer charges.

The use of selfregularizing higher order field equations is, however, problematic with respect to the fact that the quantum theory of such equations leads to indefinite metric state spaces and thus destroys the usual quantum theoretic probability interpretation. In contrast to gauge theories where indefinite metric also occurs, it is not possible to remove the indefinite metric of higher order nonlinear field equations by simply regauging them. Hence a more thorough investigation of this problem is needed.

As the state spaces of higher order nonlinear spinor field equations have to be representation spaces of the corresponding symmetry groups, it is reasonable to obtain some information about the structures of such state spaces by analyzing the corresponding representations for indefinite inner products. Such investigations were made by Schlieder [6] for representation spaces of the Poincaré group, by Stumpf and Scheerer [7] with respect to the construction of functional basis states for fermions which are to be representation spaces of the Poincaré group, and by Brunet and Kramer [8] in connection with representations of the symplectic group associated with the canonical commutation relations. Independently of group representations a functional analysis of linear indefinite inner product spaces can be developed. The results obtained in this field are collected in a book by Bognar [9].

It is, however, difficult to draw further conclusions from these general investigations for the problems of interest. This stems from the fact that for higher order nonlinear quantum fields the metrical structure of the state space is completely deter-

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mined by the dynamics of the system and cannot be postulated at will without regard to these dynamics. Hence for an investigation of the inner product space of a higher order nonlinear quantum field one should have solved its dynamical problem which itself is an unsolved mathematical task.

If one tries to get some information about the physical meaning of indefinite inner product spaces beyond a general mathematical analysis, apart from higher order field equations a lot of other very simplified models can be treated which exhibit this property. Such simplified models were extensively studied in the literature. Most of them are non-relativistic or are formulated in the interaction representation. A survey of these investigations is given in the book of Nagy [10] and the review article of Nakanishi [11]. Some recent papers in this field are cited in [3]. For many of these models it is, or it seems possible to give a consistent physical interpretation although indefinite metric is involved. Due to the nonrelativistic character or due to the interaction representation used, the results of these investigations are, however, of little use for the more general problem of the effect of indefinite metric in unified field models. In such models bound states play an important role, and other ideas must be developed to master the indefinite metric problem.

In a preceding paper [12] it was proposed that any physical particle must occur as a bound state of invisible fermions, *i.e.* that all elementary basic fermions of a unified field are not allowed to occur as free particles, *i.e.* have to be confined in bound states. If one tries to treat unified field models along these lines one has to demonstrate that confinement for the elementary fermions is possible and that the resulting bound state dynamics takes place in an ordinary Hilbert space. Apart from the completeness proofs, the latter condition implies that all bound states must exhibit a positive norm.

The attempts to verify this for simple bound states were already started in the Pauli-Villars regularized first order nonlinear spinor theory of Heisenberg [13]. By decomposing a two-fermion bound state in a free field, ghost field and dipole ghost field part, Yamazaki [14] performed a boson state normalization. By a general spectral decomposition Stumpf and Englert [15] also treated boson state normalization in nonlinear spinor theory, and Palmer and Takahashi [16] calculated the Bethe-Salpeter norm of a bound state with a multimass propagator. However, all three approaches are unsatisfactory. The first decomposition in general cannot give a complete resolution of the bound state amplitude; the second decomposition is complete, but cannot verify the norms of the components, while the Bethe-Salpeter formalism is inconsistent in inner product spaces with indefinite metric.

A correct expression for the norm of unified field states in an inner product space with indefinite metric was given in two preceding papers by the author [17]. For a long time it was, however, not possible to evaluate these general expressions properly. Recently it was shown in two papers by the author [18] that the inner products of states are connected with the left-hand and right-hand solutions of the corresponding functional equations and furthermore, that an appropriate decomposition of higher order nonlinear unified field equations is possible which allows a Schrödinger functional formulation of the theory. This functional Schrödinger representation was derived in a subsequent paper by Grosser, Hailer, Hornung, Lauxmann and Stumpf [19]. If these results are combined they make a rigorous evaluation of norm expressions etc. possible, apart, of course, from approximations which, however, do not invalidate the general procedure.

In this paper we will therefore apply these new methods to the old problem of boson normalization. In contrast to the earlier approaches we will demonstrate that, although we treat only the two-fermion sector, interesting general features of the entire theory will become obvious. Furthermore it can be seen that the calculation can be extended to higher (elementary) fermion sectors along the same lines.

Concerning confinement the simplest way to achieve this consists in assuming the ghost masses to be very large (in the limit infinitely large) so that due to energy conservation they cannot appear in physical reactions. This method has been successfully applied to Pauli-Villars regularization of the divergent expressions in gauge theories in order to properly define the renormalization procedure. Recently Dürr [20] has advocated this method for the use in unified field models. In particular, for subquark models like that of Grosser and Lauxmann [4] not only the ghost particles but all elementary fermions have to be confined. This means that real as well as ghost constituent elementary fermion fields have to be equipped with very large masses. Then it has to be demonstrated that in spite of these heavy constituent masses bound states with small masses are possible. This will be shown by treating numerically a simplified two-particle bound state model with scalar constituents. Summing up we will show that there is a close relation between confinement, hermicity and physical states which allows to develop the theory further along these lines.

1. Unified field hamiltonians

Unified field models are formulated by means of selfregularizing higher order nonlinear spinor fields. In order to concentrate on the essential properties of such models we omit all algebraic complications which are required for the formulation of realistic models and we confine ourselves to the most simple version of a higher order nonlinear spinor field equation which exhibits selfregularization, relativistic invariance and locality. Such an equation reads

$$[(-i\gamma^\mu \partial_\mu + m_1)(-i\gamma^\varrho \partial_\varrho + m_2)]_{\alpha\beta}\psi_\beta(x) = V_{\alpha\beta\gamma\delta}\psi_\beta(x)\,\bar\psi_\gamma(x)\,\psi_\delta(x)\,. \tag{1.1}$$

In order to apply the hamiltonian formalism we decompose this equation into a system of two first order nonlinear spinor field equations.

It was proved in a preceding paper [21] that the set of nonlinear equations

$$(-i\gamma^\mu \partial_\mu + m_r)_{\alpha\beta}\varphi^r_\beta(x) = \lambda_r V_{\alpha\beta\gamma\delta}\left[\sum_{r=1}^{2}\varphi^r_\beta(x)\right]\left[\sum_{r=1}^{2}\bar\varphi^r_\gamma(x)\right]\left[\sum_{r=1}^{2}\varphi^r_\delta(x)\right] \tag{1.2}$$

is connected with (1.1) by a biunique map, if this map is defined by the selfconsistent relations

$$\begin{aligned}\psi_\alpha(x) &= \varphi^1_\alpha(x) + \varphi^2_\alpha(x)\,,\\ \varphi^1_\alpha(x) &= \lambda_1(-i\gamma^\mu\partial_\mu + m_2)_{\alpha\beta}\psi_\beta(x)\,,\\ \varphi^2_\alpha(x) &= \lambda_2(-i\gamma^\mu\partial_\mu + m_1)_{\alpha\beta}\psi_\beta(x)\end{aligned} \tag{1.3}$$

with

$$\lambda_1 := (m_2 - m_1)^{-1};\quad \lambda_2 := (m_1 - m_2)^{-1}\,. \tag{1.4}$$

The Lagrangian density corresponding to equations (1.2) reads

$$\begin{aligned}L[\varphi] := {}&\frac{i}{2}\sum_{r=1}^{2}\lambda_r^{-1}[\bar\varphi^r_\alpha\gamma^\nu_{\alpha\beta}\partial_\nu\varphi^r_\beta - \partial_\nu\bar\varphi^r_\alpha\gamma^\nu_{\alpha\beta}\varphi^r_\beta - m_r\bar\varphi^r_\alpha\varphi^r_\alpha]\\ &+\frac{1}{2}V_{\alpha\beta\gamma\delta}\left[\sum_{r=1}^{2}\bar\varphi^r_\alpha\right]\left[\sum_{r=1}^{2}\varphi^r_\beta\right]\left[\sum_{r=1}^{2}\bar\varphi^r_\gamma\right]\left[\sum_{r=1}^{2}\varphi^r_\delta\right].\end{aligned} \tag{1.5}$$

and for the conjugate momenta we obtain by means of (1.5) the expressions

$$\pi^r_\alpha(x) := i\lambda_r^{-1}\varphi^r_\alpha(x)^\times;\quad r = 1, 2\,. \tag{1.6}$$

Therefrom the hamiltonian density with respect to (1.2)

$$\begin{aligned}H[\varphi] := {}&\frac{1}{2}\sum_{r=1}^{2}\lambda_r^{-1}[\bar\varphi^r_\alpha\gamma_{\alpha\beta}\cdot\nabla\varphi^r_\beta - \nabla\bar\varphi^r_\alpha\gamma_{\alpha\beta}\varphi^r_\beta + m_r\bar\varphi^r_\alpha\varphi^r_\alpha]\\ &+\frac{1}{2}V_{\alpha\beta\gamma\delta}\left[\sum_{r=1}^{2}\bar\varphi^r_\alpha\right]\left[\sum_{r=1}^{2}\varphi^r_\beta\right]\left[\sum_{r=1}^{2}\bar\varphi^r_\gamma\right]\left[\sum_{r=1}^{2}\varphi^r_\delta\right]\end{aligned} \tag{1.7}$$

results. Assuming the usual quantization by anticommutation relations between φ^r and π^r we get with (1.6)

$$[\varphi^s_\alpha(\boldsymbol r, t), \varphi^s_\beta(\boldsymbol r', t)^\times]_+ = \lambda_s\delta_{\alpha\beta}\delta(\boldsymbol r - \boldsymbol r')\,,\quad r = 1, 2\,. \tag{1.8}$$

If we assume $\lambda_1 > 0$, then it follows $\lambda_2 < 0$. Hence the anticommutator of the φ^2-field is anomalous. It is convenient to transform by the similarity transformation

$$\varphi^1(\boldsymbol r) = \lambda_1^{1/2}\Phi^1(\boldsymbol r)\,;\quad \varphi^2(\boldsymbol r) = |\lambda_2|^{1/2}\Phi^2(\boldsymbol r) \tag{1.9}$$

the anticommutators (1.8) to the standard form

$$[\Phi^1_\alpha(\boldsymbol r), \Phi^1_\beta(\boldsymbol r')]_+ = \delta_{\alpha\beta}\delta(\boldsymbol r - \boldsymbol r')\,,\quad [\Phi^2_\alpha(\boldsymbol r), \Phi^2_\beta(\boldsymbol r')]_+ = -\,\delta_{\alpha\beta}\delta(\boldsymbol r - \boldsymbol r')\,. \tag{1.10}$$

Observing $\lambda_2 = -\lambda_1$ we can rewrite the hamiltonian density (1.7) and obtain for the total hamiltonian the expression

$$H = \sum_{r=1}^{2} (-1)^{r+1} \int \Phi_\alpha^r(\mathbf{r})^\times (-i\,\alpha\cdot\nabla + \beta m_r)_{\alpha\beta} \Phi_\beta^r(\mathbf{r})\, \mathrm{d}^3 r$$
$$+ g\,\lambda_1^2 V_{\alpha\beta\gamma\delta} \int : \left[\sum_{r=1}^{2} \Phi_\alpha^r(\mathbf{r})^\times\right]\left[\sum_{r=1}^{2} \Phi_\alpha^r(\mathbf{r})\right]\left[\sum_{r=1}^{2} \Phi_\gamma^r(\mathbf{r})^\times\right]\left[\sum_{r=1}^{2} \Phi_\delta^r(\mathbf{r})\right] : \mathrm{d}^3 r. \tag{1.11}$$

This expression can be equivalently considered as the expression of a representation $\mathscr{H}$ of H in a corresponding linear state space $\mathfrak{V}$. Owing to the anomalous anticommutator in (1.10) we have to expect that $\mathfrak{V}$ is to be an inner product space with indefinite metric. In contrast to simple models which were treated in the interaction representation and which were mentioned in the introduction, the relativistic nonlinear spinor field models do not allow explicit representations of the field operators $\Phi_\alpha^r(\mathbf{r})$ and $\Phi_\alpha^r(\mathbf{r})^\times$ in terms of creation and destruction operators. Hence the investigation of the state space of the nonlinear spinor field has to be performed without the knowledge of explicit field operator representations. This topic was treated by the author and his coworkers in preceding papers and leads to the so-called functional quantum theory, some reviews of which were given in [22] and to which we refer our presentation without further comments. Nevertheless, in order to successfully apply functional quantum theory to the investigation of the state space $\mathfrak{V}$, we need further information about the behaviour of the field operators in indefinite inner product spaces. We will provide this information by means of the following assumption:

Postulate:

The operators $\Phi_\alpha^r(\mathbf{r})$ and $\Phi_\alpha^r(\mathbf{r})^\times$ in $\mathfrak{V}$ are to be hermitian conjugates, *i.e.* $\Phi_\alpha^r(\mathbf{r})^+ \equiv \Phi_\alpha^r(\mathbf{r})^\times$.

For an explicit representation of these operators in $\mathfrak{V}$, this means the following: Let $\{|m\rangle, m = 1 \dots\}$ be a complete set of linearly independent vectors of a linear metric vector space $\mathfrak{V}$, then with respect to this postulate the relation

$$\langle n | \Phi_\alpha^r(\mathbf{r})^\times | m\rangle = \langle m | \Phi_\alpha^r(\mathbf{r}) | n\rangle^\times \tag{1.12}$$

must be satisfied. It is not clear a priori whether this postulate is compatible with the general dynamics or not. It can, however, be shown that this is the case for the algebra of fermion creation and destruction operators. Therefore we assume the compatibility to hold also in our general case without further investigation.

2. Inner product representations

A relativistic theory requires a relativistic invariant description of the states $|n\rangle \in \mathfrak{V}$. In particular, we consider $|n\rangle \equiv |a\rangle$ which are to be eigenstates of interesting quantities like four-momentum etc. In the framework of functional quantum theory such states are represented by state functionals. They are defined by the following expression (for brevity spin indices are suppressed in this section)

$$|\mathfrak{T}(j,a)\rangle = \sum_{n=0}^{\infty} \sum_{\substack{k_1 \dots k_n \\ r_1 \dots r_n}} \int \tau(\overset{r_1}{x_1} k_1, \dots, \overset{r_n}{x_n} k_n | a)\, |D_n(\overset{r_1}{x_1} k_1, \dots, \overset{r_n}{x_n} k_n)\rangle\, \mathrm{d}^4 x_1 \dots \mathrm{d}^4 x_n \tag{2.1}$$

with

$$\tau(\overset{r_1}{x_1} k_1, \dots, \overset{r_n}{x_n} k_n | a) := \langle 0 | T \Phi^{r_1}(x_1 k_1) \dots \Phi^{r_n}(x_n k_n) | a\rangle\,, \tag{2.2}$$

where $|D_n\rangle$ are appropriate functional basis states and where

$$\Phi_a(x,1) := \Phi_\alpha(x)\,, \quad \Phi_\alpha(x,2) := \Phi_\alpha(x)^\times\,, \quad \textit{i.e.} \quad k = 1, 2\,.$$

Although the state functionals (2.1) are the basic quantities of the theory it is convenient to introduce the transformed state functionals

$$|\mathfrak{F}(j,a)\rangle = \exp \sum_{\substack{kk'\\ rr'}} \int j^r(x,k)\, F^{rr'}(x,k,x',k')\, j^{r'}(x',k')\, \mathrm{d}^4x\, \mathrm{d}^4x'\, |\mathfrak{T}(j,a)\rangle \tag{2.3}$$

with

$$|\mathfrak{F}(j,a)\rangle := \sum_{n=0}^{\infty} \sum_{\substack{k_1 \dots k_n\\ r_1 \dots r_n}} \int \varphi(\overset{r_1}{x_1} k_1, \dots, \overset{r_n}{x_n} k_n | a)\, |D_n(\overset{r_1}{x_1} k_1, \dots, \overset{r_n}{x_n} k_n)\rangle\, \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n \tag{2.4}$$

where the F-functions in (2.3) are the free field propagators and where (2.4) can equivalently be taken as an appropriate description of the states $|a\rangle$.

For the state functionals (2.1) as well as for the state functionals (2.4) functional state equations can be derived which are fully relativistically invariant. In spite of this complete four-dimensional characterization of the states $|a\rangle$ by their relativistically invariant state functionals (2.1) or (2.2) resp., it must be possible to determine the states $|a\rangle$ completely by considering only their projection onto a definite time $t = x_1^0, \dots x_n^0$, $n = 1 \dots \infty$ in the transition matrixelements $\tau(x_1 \dots x_n | a)$ contained in (2.1) or $\varphi(x_1 \dots x_n | a)$ contained in (2.4) resp. This follows from the fact that the system under consideration has the hamiltonian (1.11) which can be assumed to possess the states $|a\rangle$ as eigenstates in an energy representation. If the limit to equal times is performed in an appropriate way in, for instance, (2.2), it is obvious that the state $|a\rangle$ is projected onto the set of basis states $\{\langle 0|\, \Phi(\boldsymbol{r}_1 0, k_1) \dots \Phi(\boldsymbol{r}_n 0, k_n)\}$. This means that the state must allow an expansion

$$\langle a| = \sum_{n=0} \sum_{\substack{k_1 \dots k_n\\ r_1 \dots r_n}} \int \sigma(\overset{r_1}{\boldsymbol{r}_1} k_1, \dots, \overset{r_n}{\boldsymbol{r}_n} k_n | a) \langle 0|\, \Phi^{r_1}(\boldsymbol{r}_1 k_1) \dots \Phi^{r_n}(\boldsymbol{r}_n k_n)\, \mathrm{d}^3r_1 \dots \mathrm{d}^3r_n \tag{2.5}$$

with $\Phi(\boldsymbol{r}, k) \equiv \Phi(\boldsymbol{r}, 0, k)$, and from this it follows with (2.2) that the scalarproduct of two states $\langle a|$, $\langle b|$ can be written

$$\begin{aligned} \langle a | b\rangle &= \sum_{n=0}^{\infty} \sum_{\substack{k_1 \dots k_n\\ r_1 \dots r_n}} \int \sigma(\overset{r_1}{\boldsymbol{r}_1} k_1, \dots \overset{r_n}{\boldsymbol{r}_n} k_n | a) \langle 0|\, \Phi^{r_1}(\boldsymbol{r}_1 k_1) \dots \Phi^{r_n}(\boldsymbol{r}_n k_n)\, |b\rangle\, \mathrm{d}^3r_1 \dots \mathrm{d}^3r_n \\ &= \sum_{nm=0}^{\infty} \sum_{\substack{k_1 \dots k_n\\ r_1 \dots r_n}} \sum_{\substack{k_1' \dots k_n'\\ r_1' \dots r_n'}} \int \sigma(\overset{r_1}{\boldsymbol{r}_1} k_1, \dots, \overset{r_n}{\boldsymbol{r}_n} k_n | a)\, \sigma(\overset{r_1'}{\boldsymbol{r}_1'} k_1', \dots, \overset{r_n'}{\boldsymbol{r}_m'} k_m' | b)^{\times} \\ &\quad \cdot \langle 0|\, \Phi^{r_1}(\boldsymbol{r}_1 k_1) \dots \Phi^{r_n}(\boldsymbol{r}_n k_n) [\Phi^{r_1'}(\boldsymbol{r}_1' k_1') \dots \Phi^{r_m'}(\boldsymbol{r}_m' k_m')]^+ |0\rangle\, \mathrm{d}^3r_1 \dots \mathrm{d}^3r_m' . \end{aligned} \tag{2.6}$$

In the latter expression, obviously the metrical tensor $\mathfrak{G}$ of the basis system of the expansion (2.5) occurs. Due to the anomalous anticommutation relations (1.10) we may assume that this tensor characterizes an indefinite metric state space $\mathfrak{V}$ even if we are not able to give an explicit representation of the operators immediately. According to our (compatible) postulate the operators Φ^r, $\Phi^{r\times}$ must then satisfy the relation $\Phi^{r+} \equiv \Phi^{r\times}$ in $\mathfrak{V}$. From this it follows that with

$$\lim_{t_1 > \dots > t_n \to 0} \tau(\overset{r_1}{x_1} k_1, \dots, \overset{r_n}{x_n} k_n | b) = \tau(\overset{r_1}{\boldsymbol{r}_1} k_1, \dots, \overset{r_n}{\boldsymbol{r}_n} k_n | b) \tag{2.7}$$

we have

$$\tau(\overset{r_1}{\boldsymbol{r}_1} k_1, \dots, \overset{r_n}{\boldsymbol{r}_n} k_n | b) \equiv \langle 0|\, \Phi^{r_1}(\boldsymbol{r}_1 k_1) \dots \Phi^{r_n}(\boldsymbol{r}_n k_n)\, |b\rangle \tag{2.8}$$

and

$$\mathfrak{G} = \{\langle 0|\, \Phi^{r_1}(\boldsymbol{r}_1 k_1) \dots \Phi^{r_n}(\boldsymbol{r}_n k_n)\, \Phi^{r_m'}(\boldsymbol{r}_m' k_m')^+ \dots \Phi^{r_1'}(\boldsymbol{r}_1' k_1')^+ |0\rangle,\ n, m = 1 \dots\} . \tag{2.9}$$

For further investigation we additionally need the set of dual basis states $\{|\overset{r'_1}{\boldsymbol{r}'_1} k'_1, \dots, \overset{r'_m}{\boldsymbol{r}'_m} k'_m\rangle\}$ which is defined by the relations

$$\langle 0 | \Phi^{r_1}(\boldsymbol{r}_1 k_1) \dots \Phi^{r_n}(\boldsymbol{r}_n k_n) | \overset{r'_1}{\boldsymbol{r}'_1} k'_1, \dots, \overset{r'_m}{\boldsymbol{r}'_m} k'_m\rangle$$

$$= \delta_{nm} \sum_{\lambda_1 \dots \lambda_n} (-1)^{\mathrm{P}} \delta_{r'_1 r_{\lambda_1}} \delta(\boldsymbol{r}'_1 - \boldsymbol{r}_{\lambda_1}) \dots \delta_{r'_n r_{\lambda_n}} \delta(\boldsymbol{r}'_n - \boldsymbol{r}_{\lambda_n}) . \tag{2.10}$$

As long as no explicit representations of the field operators are known it is impossible to calculate this dual set from (2.10) really. Nevertheless it is a valuable tool for further investigation.

We expand a state $|b\rangle$ in terms of the dual states and obtain

$$|b\rangle = \sum_{n=0}^{\infty} \sum_{\substack{k_1 \dots k_n \\ r_1 \dots r_n}} \int \tau(\overset{r_1}{\boldsymbol{r}_1} k_1, \dots, \overset{r_n}{\boldsymbol{r}_n} k_n | b) \, |\overset{r_1}{\boldsymbol{r}_1} k_1, \dots, \overset{r_n}{\boldsymbol{r}_n} k_n\rangle \, \mathrm{d}^3 r_1, \dots, \mathrm{d}^3 r_n . \tag{2.11}$$

By multiplication with the original basis states from the left it can easily be verified that the functions (2.8) occur as expansion coefficients in (2.11).

The properties of eigenvalue problems formulated by means of the expansions (2.5) and (2.11) resp., were studied in a preceding paper [23] for the case of positive metric. Therefore we have only to generalize these considerations to the case of an indefinite state space. We denote the set of expansion coefficients in a vector notation by

$$\mathfrak{s}(a) := \{\sigma_n(a), n = 1 \dots \infty\} \quad \text{and}$$
$$\mathfrak{t}(\alpha) := \{\tau_n(a), n = 1 \dots \infty\}$$

and for instance consider the eigenvalue problem of H in $\mathfrak{V}$. Then if we observe that the dual states possess the metrical tensor $\tilde{\mathfrak{G}}$ we have for (2.11) the eigenvalue equation

$$\tilde{\mathscr{H}} \cdot \mathfrak{t}(b) = E_b \tilde{\mathfrak{G}} \cdot \mathfrak{t}(b) \tag{2.12}$$

if we project the eigenvalue equation $H|b\rangle = E_b|b\rangle$ on dual basis states. On the other hand, we can project this equation on the original basis states and obtain

$$\tilde{\hat{\mathscr{H}}} \cdot \mathfrak{t}(b) = E_b \mathfrak{t}(b) . \tag{2.13}$$

Hence it must be $\tilde{\mathscr{H}} = \tilde{\mathfrak{G}} \tilde{\hat{\mathscr{H}}}$. Similarly we obtain for the eigenvalue equation $\langle a | H = E_a \langle a |$ with (2.5) two representations, namely

$$\mathfrak{s}(a) \cdot \mathscr{H} = E_a \mathfrak{s}(a) \cdot \mathfrak{G} \tag{2.14}$$

and

$$\mathfrak{s}(a) \cdot \hat{\mathscr{H}} = E_a \mathfrak{s}(a) \tag{2.15}$$

with $\mathscr{H} = \hat{\mathscr{H}} \mathfrak{G}$. Although both types of equations are derived in $\mathfrak{V}$ the mixed co-contravariant representations (2.13) and (2.15) can formally be interpreted as a non-selfadjoint representation in a positive state space $\mathfrak{H}$. However, such an interpretation becomes only meaningful if one is able to relate both representations. The following theorem holds:

Theorem 2.1. Let $\tilde{\mathscr{H}}$ and $\mathscr{H}$ be dual representations of a hamiltonian H in $\mathfrak{V}$. Then for the mixed co-contravariant representations $\tilde{\hat{\mathscr{H}}}$ and $\hat{\mathscr{H}}$ it is $\tilde{\hat{\mathscr{H}}} \equiv \hat{\mathscr{H}}$, and $\mathfrak{t}(a)$ and $\mathfrak{s}(a)$ are the right-hand and left-hand solutions of the same equation for a definite eigenvalue E_a.

Proof: We introduce the abbreviations

$$\langle \xi_n | := \langle 0 | \Phi(\xi_1) \dots \Phi(\xi_n) ,$$
$$|\xi_n\rangle := [\Phi(\xi_1) \dots \Phi(\xi_n)]^+ |0\rangle \tag{2.16}$$

for the original basis vector system and

$$\langle \xi^n | := \langle \xi_1 \dots \xi_n | ,$$
$$|\xi^n\rangle := |\xi_1 \dots \xi_n\rangle = (\langle \xi_1 \dots \xi_n |)^+ \tag{2.17}$$

for the dual basis vector system. Then it is

$$\mathfrak{G} \equiv g_{nm} := \langle \xi_n | \xi_m \rangle ,$$
$$\tilde{\mathfrak{G}} \equiv g^{nm} := \langle \xi^n | \xi^m \rangle . \tag{2.18}$$

It follows from the general theory of linear spaces that $\tilde{\mathfrak{G}} \equiv \mathfrak{G}^{-1}$. Furthermore we have

$$\tilde{\mathscr{H}} \equiv H^{nm} := \langle \xi^n | H | \xi^m \rangle \tag{2.19}$$

and

$$\mathscr{H} \equiv H_{nm} := \langle \xi_n | H | \xi_m \rangle . \tag{2.20}$$

From the relations $\tilde{\mathscr{H}} = \tilde{\mathfrak{G}} \tilde{\hat{\mathscr{H}}}$ and $\mathscr{H} = \hat{\mathscr{H}} \mathfrak{G}$ we therefore obtain

$$\tilde{\hat{\mathscr{H}}} = \mathfrak{G} \tilde{\mathscr{H}}, \quad \hat{\mathscr{H}} = \mathscr{H} \mathfrak{G}^{-1} . \tag{2.21}$$

From the general theory of linear spaces it also follows that

$$\tilde{\mathscr{H}} = \mathfrak{G}^{-1}\mathscr{H}\mathfrak{G}^{-1} \tag{2.22}$$

holds. Hence if we substitute (2.22) into (2.21) we obtain

$$\hat{\tilde{\mathscr{H}}} = \mathfrak{G}\mathfrak{G}^{-1}\mathscr{H}\mathfrak{G}^{-1} = \mathscr{H}\mathfrak{G}^{-1} = \hat{\mathscr{H}} \tag{2.23}$$

and with (2.13) and (2.15) the proposition. Q.E.D.

In [23] this theorem was proved for selfadjoint operators in Hilbert spaces $\mathfrak{H}$ by means of spectral decompositions of these operators. Such spectral decompositions in general do not exist in linear spaces $\mathfrak{V}$ with an indefinite inner product. Nevertheless, the above discussion shows that the connection between the inner product of physical states (2.6) which can be written with (2.8) and our further abbreviations as

$$\langle a \mid b \rangle = \mathfrak{s}(a) \cdot \mathfrak{t}(b) \tag{2.24}$$

and the right-hand and left-hand solutions of the equations

$$\mathfrak{s}(a) \cdot \hat{\mathscr{H}} = E_a \mathfrak{s}(a)\,; \quad \hat{\mathscr{H}} \cdot \mathfrak{t}(b) = E_b \mathfrak{t}(b) \tag{2.25}$$

can be also maintained in the case of indefinite inner product spaces. Therefore the question arises where differences between $\mathfrak{H}$ and $\mathfrak{V}$-spaces can be seen. Before discussing this problem we derive a further theorem:

Theorem 2.2: Let $\mathscr{H}$ be a hermitian representation of H in $\mathfrak{V}$. Then for real non-degenerate eigenvalues $\{E_a^r\}$ the corresponding eigenstates $\{|a, r\rangle\}$ in $\mathfrak{V}$ are orthogonal $\langle a, r \mid b, r\rangle = 0$ for $a \neq b$, while for complex non-degenerate eigenvalues $\{E_a^c\}$ the norm of the corresponding eigenstates $\{|a, c\rangle\}$ vanishes, *i.e.* $\langle a, c \mid a, c\rangle = 0$.

Proof: According to our postulate we have $\Phi(\xi)^\times \equiv \Phi(\xi)^+$ in V. Therefore in the basis system (2.16) it can be easily seen that (1.11) leads to a hermitian representation $\mathscr{H}$, given by (2.20). As due to this postulate $\mathfrak{G}$ and $\mathfrak{G}^{-1}$ must also be hermitian, it follows that $\tilde{\mathscr{H}}$ given by (2.19) must be hermitian, too. Therefore the equations (2.12) and (2.14) have hermitian matrix representations. From this it follows that

$$\mathfrak{t}(b)^\times \cdot \tilde{\mathscr{H}}^+ = \mathfrak{t}(b)^\times \cdot \tilde{\mathscr{H}} = E_b \mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1} \tag{2.26}$$

and

$$\mathscr{H}^+ \cdot \mathfrak{s}(a)^\times = \mathscr{H} \cdot \mathfrak{s}(a)^\times = E_a \mathfrak{G} \cdot \mathfrak{s}(a)^\times \tag{2.27}$$

holds. This yields in the usual way for real eigenvalues

$$\mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1} \cdot \mathfrak{t}(a) = \mathfrak{s}(b) \cdot \mathfrak{G} \cdot \mathfrak{s}(a)^\times = 0 \tag{2.28}$$

and by means of the representations (2.5) and (2.11) and the definitions (2.18), this is equivalent to $\langle a \mid b\rangle = 0$. Similar conclusion can be drawn with respect to the complex eigenvalues. Q.E.D.

By comparison with (2.24) we conclude that

$$\mathfrak{t}(a) = \mathfrak{G} \cdot \mathfrak{s}(a)^\times\,; \quad \mathfrak{s}(b) = \mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1}\,. \tag{2.29}$$

It can easily be verified that these relations really hold.

Theorem 2.3: The relations (2.29) are satisfied for non-degenerate right-hand and left-hand solutions $\mathfrak{s}(a)$ and $\mathfrak{t}(a)$ of (2.25).

Proof: With (2.25) the (2.12) and (2.14) are satisfied, too. From these equations (2.26) and (2.27) follow. We consider (2.26). By means of (2.22) it can be rewritten into

$$\mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1}\mathscr{H}\mathfrak{G}^{-1} = E_b \mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1} \tag{2.30}$$

or

$$\mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1}\mathscr{H} = E_b \mathfrak{t}(b)^\times = E_b \mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1}\mathfrak{G}\,. \tag{2.31}$$

Therefore $\mathfrak{t}(b)^\times \cdot \mathfrak{G}^{-1}$ must be a solution of equation (2.14). As the solutions are assumed to be non-degenerate, relation (2.29) must hold. Q.E.D.

Furthermore the following theorem holds:

Theorem 2.4: The mixed representation $\hat{\mathscr{H}}$ of a hermitian hamiltonian H in $\mathfrak{V}$ is in general a non-hermitian matrix, *i.e.* $\hat{\mathscr{H}} \neq \hat{\mathscr{H}}^+$, in particular it is $\hat{\mathscr{H}}^+ = \mathfrak{G}^{-1}\hat{\mathscr{H}}\mathfrak{G}$.

Proof: According to theorem 2.2 the representation $\mathscr{H}$ of H in $\mathfrak{V}$ is hermitian. From (2.21) we therefore have $\mathscr{H} = \hat{\mathscr{H}}\mathfrak{G} = \mathscr{H}^+ = (\hat{\mathscr{H}}\mathfrak{G})^+ = \mathfrak{G}\hat{\mathscr{H}}^+$ and thus $\hat{\mathscr{H}}^+ = \mathfrak{G}^{-1}\hat{\mathscr{H}}\mathfrak{G} \neq \hat{\mathscr{H}}$. Q.E.D.

In linear algebra it is shown that problems of the type (2.25) can possess real as well as complex eigenvalues. Furthermore, in a nontrivial indefinite metric space $\mathfrak{V}$ no transformations to orthonormal basis systems can be performed. Hence the difference between $\mathfrak{H}$ and $\mathfrak{V}$ manifests itself in the occurrence of complex eigenvalues and in the possibility that the norm expressions $\langle a \mid a\rangle$ even of real eigenvalues may disappear or become negative. Treating a field theory in $\mathfrak{V}$ one has to demonstrate that in spite of these drawbacks a physical and probabilistic interpretation is possible. This will be done for the two-fermion sector of our model in Part II. Additionally, it should be noted: Theorem 2.2 is well known in linear analysis, cf. Nakanishi [11]. We have it formulated only in a suitable way with respect to our deduction. Furthermore: Left-hand and right-hand solutions of unsymmetric operators were introduced in the early development of the theory of integral equations, cf. Schmeidler [24].

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